Financial Markets I

Lecture 5: Portfolio Theory

Master Finance & Strategy

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SciencesPo

Overview of Lecture 5

Big question: How to choose a portfolio of stocks?

1. Return on a Portfolio.
2. Expected Return and Variance of a Portfolio.
3. Benefits of Diversification.
4. Mean-Variance Optimization and the Portfolio Frontier.
5. Portfolio Frontier with Risky Assets Only.
6. Portfolio Frontier with a Riskless Asset: the CML.
7. Effects of Small Changes in Portfolio Composition.
8. Important Property of Portfolios on the CML.

1. Return on a Portfolio

- Consider a portfolio consisting of X_1 dollars invested in Disney and X_2 dollars invested in IBM.
- The value of the portfolio at date 0 is $X = X_1 + X_2$.
- The value of the portfolio at date 1 is

$$X_1(1 + R_1) + X_2(1 + R_2),$$

  where R_1 is the rate of return on Disney and R_2 the rate of return on IBM between dates 0 and 1.

- The return on the portfolio is

$$\begin{aligned} R &= \frac{X_1(1 + R_1) + X_2(1 + R_2) - X}{X} \\ &= \frac{X_1}{X} R_1 + \frac{X_2}{X} R_2. \end{aligned}$$

Portfolio Weights

- The **portfolio weights**

$$w_1 = \frac{X_1}{X} \quad \text{and} \quad w_2 = \frac{X_2}{X}$$

  give the fraction of total portfolio value invested in each stock. They sum to 1.

- The return on the portfolio is

$$R = w_1 R_1 + w_2 R_2.$$

  It is a **weighted average** of the returns on the individual stocks.

- More generally, the return on a portfolio with N stocks is

$$R = \sum_{n=1}^{N} w_n R_n, \qquad \text{where} \quad w_n = \frac{X_n}{X}.$$

Example 1

Compute the return on a portfolio consisting of \$300 in Disney and \$100 in IBM.

- We have

$$w_1 = \frac{300}{400} = 0.75$$

and

$$w_2 = \frac{100}{400} = 0.25.$$

- The return on the portfolio is

$$R = 0.75 \times R_1 + 0.25 \times R_2.$$

Example 2: Portfolio Return with Short Sales

Start with the same amount of initial capital, \$400. Instead of buying \$100 of IBM, we *'short sell'* \$100. What is the return on the portfolio?

- A **short sale** consists of selling a stock that we do not own.
- Steps:
  - Date 0: Borrow the stock from a broker. Sell the stock in the market.
  - Date 1: Buy the stock in the market. Return the stock to the broker.

  (In case the stock pays a dividend between dates 0 and 1, you need to compensate the broker for the amount of dividend payment)
- A short sale is profitable if the stock price goes down.

Example 2 (cont'd)

- $t = 0$: Get \$100 from selling IBM. Invest \$100+\$400=\$500 in Disney.
- $t = 1$: Disney is worth \$500$(1 + R_1)$. Need to pay \$100$(1 + R_2)$ to close the short IBM position. Net portfolio value is

$$500(1 + R_1) - 100(1 + R_2).$$

- Return on the portfolio is

$$\begin{aligned} R &= \frac{500(1 + R_1) - 100(1 + R_2) - 400}{400} \\ &= \frac{500}{400} R_1 - \frac{100}{400} R_2 \\ &= 1.25 \times R_1 - 0.25 \times R_2. \end{aligned}$$

- Portfolio return is still a weighted average of returns on the individual stocks. Stocks that are sold short have negative weight. Weights still sum to 1.

2. Expected Return and Variance of a Portfolio

- The **expected return of a portfolio** with N stocks is

$$E(R) = \sum_{n=1}^{N} w_n E(R_n).$$

- It is a weighted average of the expected returns of the individual stocks.
- In practice, we do not know the expected returns of the individual stocks, but can estimate them using sample averages.

Expected Return on a Portfolio: Example

Compute (an estimate of) the expected return of a portfolio consisting of \$300 in Disney and \$100 in IBM.

- We have

$$w_1 = \frac{300}{400} = 0.75$$

and

$$w_2 = \frac{100}{400} = 0.25.$$

- Estimating the expected returns by the sample average for 1967-2017, we found in Lecture 4

$$E(R_1) = 19.1\%, \qquad E(R_2) = 11.7\%.$$

- The expected return of the portfolio is

$$E(R) = 0.75 \times E(R_1) + 0.25 \times E(R_2) = 17.25\%.$$

Variance and Standard Deviation

- The **variance of a portfolio** depends not only on the variances of the individual stocks, but also on their covariances.
- It is greater when the covariances are positive rather than negative.
- The variance of a portfolio with two stocks is

$$V(R) = w_1^2 V(R_1) + w_2^2 V(R_2) + 2 w_1 w_2 Cov(R_1, R_2).$$

- More generally, the variance of a portfolio with N stocks is

$$V(R) = \sum_{n=1}^{N} w_n^2 V(R_n) + 2 \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} w_n w_m Cov(R_n, R_m).$$

- The **portfolio standard deviation** is

$$\sigma(R) = \sqrt{V(R)}.$$

Properties of Covariance Operator and Portfolio Variance Formula

- Given random variables X, Y, Z and constants a and b, we have

$$V(X) = Cov(X, X)$$
$$Cov(aX, bY) = (ab)Cov(X, Y)$$
$$Cov(X + Y, Z) = Cov(X, Z) + Cov(Y, Z).$$

- The variance of a portfolio with N stocks is

$$V(R) = V\left(\sum_{n=1}^{N} w_n R_n\right) = Cov\left(\sum_{n=1}^{N} w_n R_n, \sum_{m=1}^{N} w_m R_m\right) = \sum_{n=1}^{N}\sum_{m=1}^{N} w_n w_m Cov(R_n, R_m).$$

- The formula given on previous slide follows from the fact that

$$Cov(R_n, R_n) = V(R_n) \quad \text{and} \quad Cov(R_n, R_m) = Cov(R_m, R_n).$$

Variance and Standard Deviation (cont'd)

- The previous equations can be written in terms of individual standard deviations and pairwise correlations, rather than variances and covariances.
  - Replace $V(R_n)$ by $\sigma(R_n)^2$
  - Replace $Cov(R_n, R_m)$ by $\rho(R_n, R_m)\sigma(R_n)\sigma(R_m)$.

- For instance, the standard deviation of a portfolio of two stocks is given by

$$\sigma(R) = \sqrt{w_1^2\sigma(R_1)^2 + w_2^2\sigma(R_2)^2 + 2w_1 w_2\rho(R_1, R_2)\sigma(R_1)\sigma(R_2)}.$$

- In practice, we do not know the standard deviations and correlations of the individual stocks, but can estimate them using sample standard deviations and correlations.

Portfolio Standard Deviation: Example

Compute (an estimate of) the standard deviation of a portfolio consisting of \$300 in Disney and \$100 in IBM.

- The portfolio weights are $w_1 = 0.75$ and $w_2 = 0.25$.
- Estimating the standard deviations and correlation by their sample estimates for 1967-2017 data, we found

$$\sigma(R_1) = 39.8\%, \quad \sigma(R_2) = 28.2\%, \quad \text{and} \;\; \rho(R_1, R_2) = 0.06.$$

- The standard deviation of the portfolio is given by

$$\begin{aligned} \sigma(R)^2 &= (0.75)^2\sigma(R_1)^2 + (0.25)^2\sigma(R_2)^2 \\ &\quad + 2(0.75)(0.25)\rho(R_1, R_2)\sigma(R_1)\sigma(R_2) \\ \Rightarrow \quad \sigma(R) &= 31.1\%. \end{aligned}$$

3. Benefits of Diversification

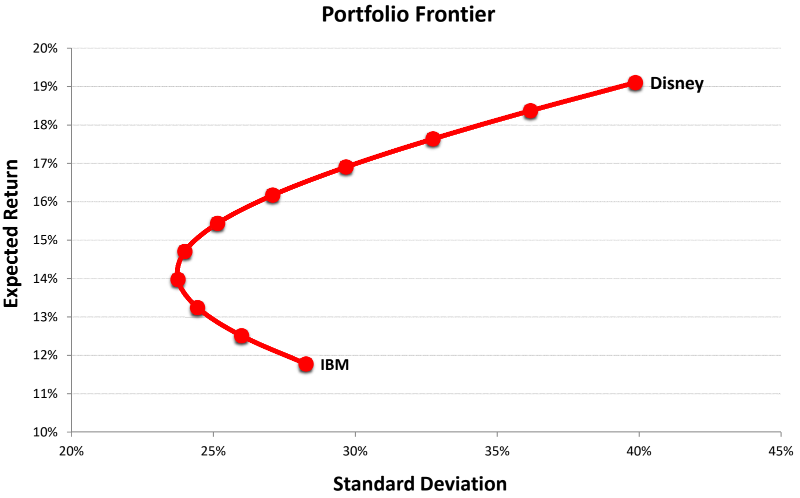
The table below represents the expected return and standard deviation of a Disney/IBM portfolio, as we vary the weight, w_1, on Disney.

w_1	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$E(R)$	11.7	12.5	13.2	13.9	14.7	15.4	16.1	16.9	17.6	18.3	19.1
$\sigma(R)$	28.2	25.9	24.4	23.7	23.9	25.1	27.1	29.6	32.7	36.1	39.8

What do you notice?

A Useful Graph

This graph represents combinations of portfolio standard deviation and expected return that we can achieve for different portfolio weights.



Benefits of Diversification

- The key observation from the table and the graph is that diversification can reduce risk substantially.
  - It is possible to form portfolios that have a lower standard deviation than a pure Disney or IBM portfolio.
- A diversified portfolio can have lower risk because the individual stocks do not always move together.
- A second observation from the table and the graph is that diversification does not necessarily reduce expected return.
  - Starting from a pure IBM portfolio, we can both reduce volatility and raise expected return by putting some weight on Disney.

4. Portfolio Optimization

- We now get to the big question: How to choose a stock portfolio?
- Our analysis assumes that we care **only** about portfolio mean and variance. We like high expected returns and dislike volatility.
  - This is referred to as "mean-variance" portfolio analysis.
- Step 1: Among all portfolios that have a given expected return, which is the portfolio with the minimum variance (or standard deviation)?
  - Step 1 will give us a set of portfolios, one for each target level of expected return.
  - This set is the **portfolio frontier** (PF). Its elements are the **frontier portfolios**. We can restrict our attention to these portfolios.
- Step 2: Which is the best portfolio on the PF?
  - The answer to Step 2 depends on how we trade off risk and return.

Application: Global Portfolio Allocation

- To illustrate **mean-variance optimization**, we consider the problem of choosing a global equity portfolio using historical data.
- In place of individual stocks, we use country indices from seven large stock markets. These indices are constructed by Morgan Stanley Capital International (MSCI).
- The **inputs** are estimates of means, standard deviations and correlations for country indices.
- Using returns from 1970-2008, we compute:

	Canada	China	France	HK	Japan	UK	US
Means	13.2	6.5	14.4	23.7	13.2	14.0	11.4
St. Devs.	19.0	38.1	22.0	36.2	21.8	22.2	15.1

Global Portfolio Allocation (cont'd)

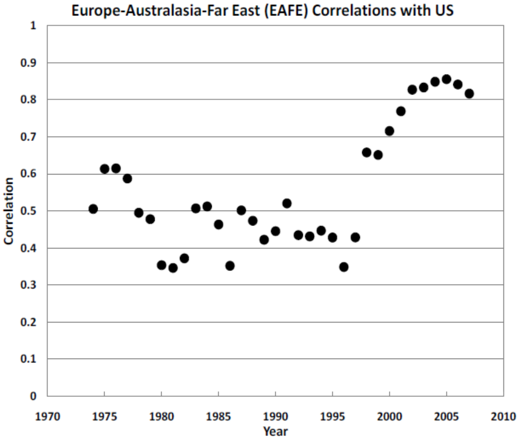
- Sample correlations 1970-2008, based on monthly returns

	CAN	CHN	FRA	HK	JAP	UK	US
CAN	1						
CHN	0.49	1					
FRA	0.48	0.31	1				
HK	0.38	0.65	0.31	1			
JAP	0.33	0.17	0.40	0.31	1		
UK	0.52	0.38	0.58	0.39	0.37	1	
US	0.71	0.43	0.50	0.36	0.31	0.54	1

- We should keep in mind that these numbers are only estimates.
  - The result of mean-variance optimization is only as good as our estimates are: **garbage in, garbage out.**

Caveat: Estimation Error

Using monthly returns over 5-year rolling windows (i.e., for the 5 years preceding a given year), we see that correlation estimates vary with the estimation window. It is hard to tell apart genuine time variation in correlation from estimation error.



5. Portfolio Frontier with Risky Assets Only

- **Optimization Problem:** Among all portfolios that have a given target expected return equal to μ, which is the portfolio with the minimum variance?
- For a given μ, we want to choose portfolio weights $w_1, \ldots, w_N$, to minimize

$$V(R) = \sum_{n=1}^{N} w_n^2 V(R_n) + 2 \sum_{n=1}^{N-1} \sum_{m>n} w_n w_m Cov(R_n, R_m)$$

subject to

$$\sum_{n=1}^{N} w_n = 1$$

and

$$E(R) = \sum_{n=1}^{N} w_n E(R_n) = \mu.$$

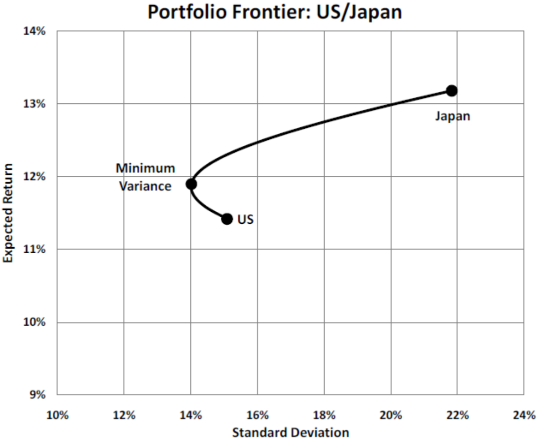
Portfolio Frontier with Risky Assets Only (cont'd)

- Solving this problem for different values of the **target expected return** μ will give us the portfolio frontier and the frontier portfolios.

- Application: We will construct portfolio frontiers in 3 cases
  - US and Japan only, without short sales.
  - US and Japan only, with short sales.
  - all seven countries with short sales (global portfolio frontier).

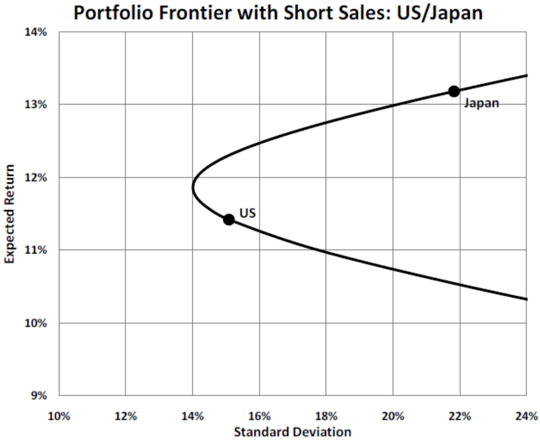
Portfolio Frontier with Two Assets

With two assets, the PF can be determined very easily. Any portfolio is a frontier portfolio because there is no other portfolio having the same expected return (this is no longer true with $N > 2$ assets).



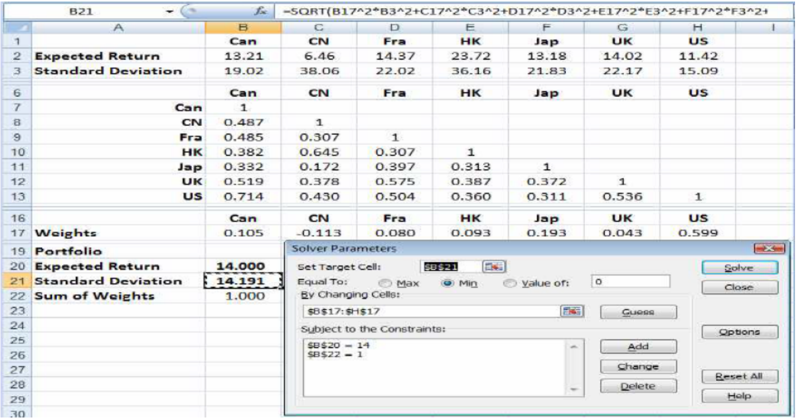
Frontier with Two Assets and Short Sales

Short sales expand the PF. Expected returns above Japan or below US can be achieved. PF becomes a hyperbola.



Portfolio Frontier with More than Two Assets

- With more than two assets, not all portfolios are frontier portfolios.
- To determine a frontier portfolio, we need to solve the optimization problem. This can be done in Excel with `Solver`.

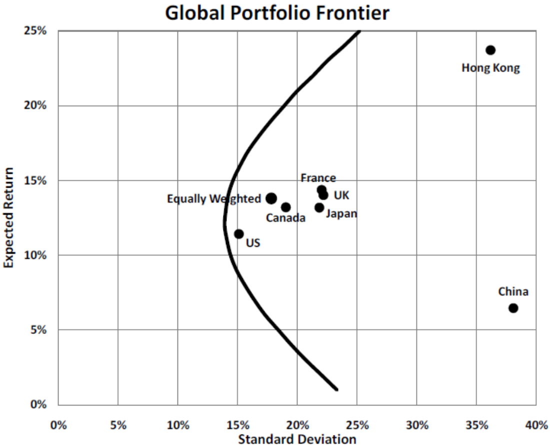


B21 =SQRT(B17^2*B3^2+C17^2*C3^2+D17^2*D3^2+E17^2*E3^2+F17^2*F3^2+

	A	B	C	D	E	F	G	H
1		Can	CN	Fra	HK	Jap	UK	US
2	Expected Return	13.21	6.46	14.37	23.72	13.18	14.02	11.42
3	Standard Deviation	19.02	38.06	22.02	36.16	21.83	22.17	15.09
6		Can	CN	Fra	HK	Jap	UK	US
7	Can	1						
8	CN	0.487	1					
9	Fra	0.485	0.307	1				
10	HK	0.382	0.645	0.307	1			
11	Jap	0.332	0.172	0.397	0.313	1		
12	UK	0.519	0.378	0.575	0.387	0.372	1	
13	US	0.714	0.430	0.504	0.360	0.311	0.536	1
16		Can	CN	Fra	HK	Jap	UK	US
17	Weights	0.105	-0.113	0.080	0.093	0.193	0.043	0.599
19	Portfolio							
20	Expected Return	14.000						
21	Standard Deviation	14.191						
22	Sum of Weights	1.000						

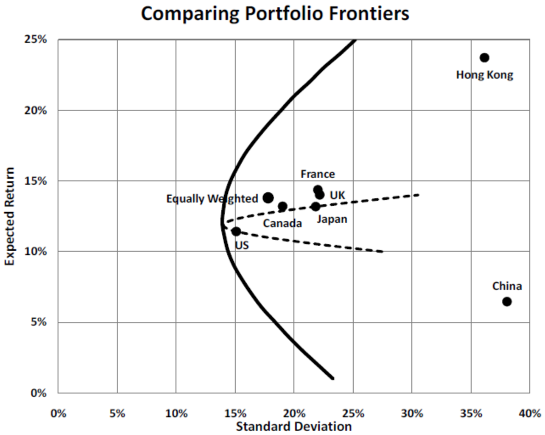
Portfolio Frontier of All Countries

Not all portfolios are frontier portfolios. In particular, observe that none of the individual countries, nor the equally weighted portfolio, are on the PF.



Portfolio Frontier: Diversification Benefits

Compare PF for 2 vs 7 countries. Adding assets shifts the PF to the left. The variance that can be achieved for a given expected return decreases.



Scope for Diversification

- By adding assets to the 'investment opportunity set', we can generally reduce the variance that can be achieved for a given expected return.
- This is simply another way to say that diversification reduces risk.
- To what extent can risk be reduced?
  - Can risk be reduced to zero, by adding very many assets?
  - Answer: No, because of systematic risk.

Systematic vs. Idiosyncratic Risk

Consider a group of assets.

- **Systematic Risk** refers to risk which affects all assets.
  - If, for instance, the assets are US stocks, systematic risk corresponds to events affecting the US economy.
- **Idiosyncratic Risk** refers to risk which affects only one asset.
  - For US stocks, idiosyncratic risk corresponds to events affecting only one particular company or industry.
- Diversification within the group of assets reduces, and eventually eliminates, idiosyncratic risk.
- However, it cannot reduce systematic risk.
- Diversification outside the group of assets (if it is possible) is more effective in reducing risk.

6. Portfolio Frontier with a Riskless Asset

- Now suppose that we can *also* save or borrow at riskfree rate R_f.
  - Variance of riskless asset, and covariance with all risky assets, is 0.
- To start with, consider a portfolio combining positions in the riskless asset and in <u>one</u> risky asset with rate of return R_r, with $E(R_r) > R_f$.
- Let w denote the weight on risky asset. The portfolio return is

$$R = wR_r + (1-w)R_f = R_f + w(R_r - R_f).$$

- The expected return is

$$\begin{aligned} E(R) &= wE(R_r) + (1-w)R_f \\ &= R_f + w[E(R_r) - R_f]. \end{aligned}$$

- The variance and standard deviation of the portfolio are

$$V(R) = w^2 V(R_r), \quad \text{and} \quad \sigma(R) = w\sigma(R_r).$$

One Risky Asset and a Riskless Asset

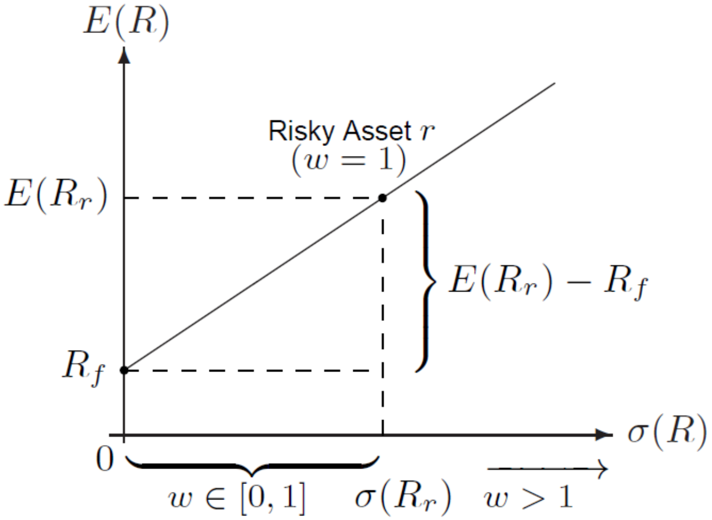
- What combinations of expected return and standard deviation can we get by taking positions in the riskless asset and in the risky asset?
- Any level of volatility $\sigma(R)$ can be achieved with weight $w = \frac{\sigma(R)}{\sigma(R_r)}$ on the risky asset. The expected return on this portfolio is

$$E(R) = R_f + \frac{\sigma(R)}{\sigma(R_r)}[E(R_r) - R_f] = R_f + \frac{E(R_r) - R_f}{\sigma(R_r)}\sigma(R).$$

- The relation between expected return and standard deviation is **affine** (see graphical representation on the next slide).
- The slope of the line is given by the **Sharpe ratio** of the risky asset

$$\frac{E(R_r) - R_f}{\sigma(R_r)}.$$

One Risky Asset and a Riskless Asset: Illustration



Multiple Risky Assets and a Riskless Asset

- Consider a portfolio with positions in N risky assets and the riskless asset, with weights $\{w_n\}_{n=1}^N$, and w_f such that $w_f = 1 - \sum_{n=1}^N w_n$.
- The return on the portfolio is

$$R = \sum_{n=1}^{N} w_n R_n + w_f R_f = R_f + \sum_{n=1}^{N} w_n (R_n - R_f).$$

- The expected return of the portfolio is

$$E(R) = R_f + \sum_{n=1}^{N} w_n [E(R_n) - R_f].$$

- The variance $V(R)$ of the portfolio is equal to

$$V\left(\sum_{n=1}^{N} w_n R_n\right) = \sum_{n=1}^{N} w_n^2 V(R_n) + 2 \sum_{n=1}^{N-1} \sum_{m>n} w_n w_m Cov(R_n, R_m).$$

Frontier Porfolios

- **Optimization problem:** Given some target expected return μ, choose <u>risky portfolio weights</u> $\{w_n\}_{n=1}^N$ to minimize

$$V(R) = \sum_{n=1}^{N} w_n^2 V(R_n) + 2 \sum_{n=1}^{N-1} \sum_{m>n} w_n w_m Cov(R_n, R_m)$$

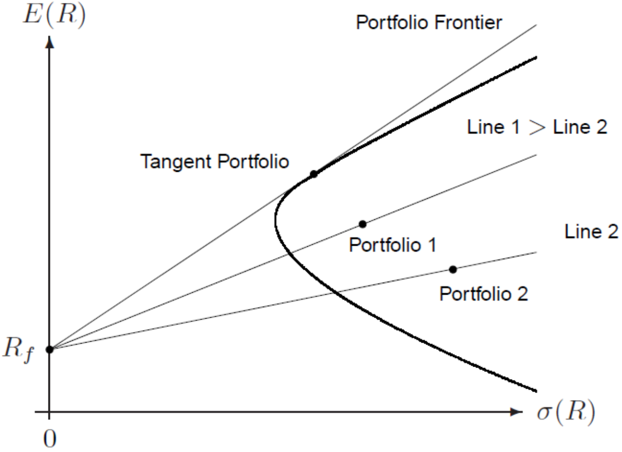
subject to

$$E(R) = R_f + \sum_{n=1}^{N} w_n [E(R_n) - R_f] = \mu.$$

- Note that we only need to choose the weights of the risky assets. The weight of the riskless asset is obtained as $w_f = 1 - \sum_{n=1}^{N} w_n$.
    - A negative weight $w_f < 0$ means that we take leverage.
- What does the portfolio frontier look like?

PF with a Riskless Asset

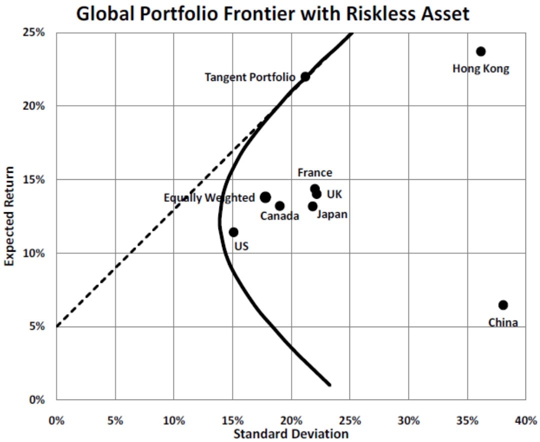
Consider the lines linking the riskless asset with portfolios inside and on the hyperbola. The portfolio frontier is the line with the steepest slope.



Properties of the Portfolio Frontier

- The PF is the line linking riskless asset with **tangent portfolio** (TP).
  - It is sometimes called the **Capital Market Line**.
- All frontier portfolios can be viewed as combined positions in the riskless asset and the tangent portfolio.
- The portfolios below the TP involve a positive weight on the riskless asset (lending).
- The portfolios above the TP involve a negative weight (borrowing).
- To construct the PF, we only need to
  - determine one frontier portfolio by solving the optimization problem once for a given target expected return $\mu > R_f$;
  - draw the line linking that portfolio to the riskless asset.

Application



The figure assumes the rate of return on the riskless asset is 5%.

Which Portfolio to Choose?

- We have seen how to construct the PF and determine the composition of the frontier portfolios.
- Assuming that we care only about mean and variance, we should choose a portfolio on the PF.
- But then comes step 2: which frontier portfolio should we choose?
- Short answer: it depends on how we trade off risk vs return, i.e., on our **risk aversion**.
  - If we are very risk-averse, we should choose a portfolio closer to the riskless asset.
  - If we are not very risk-averse, we should choose a portfolio closer to the TP, and even above the TP.

7. Effects of a Small Change in Composition

- Consider a portfolio P with risky weights $\{w_n\}_{n=1}^N$. Suppose that we *slightly* alter the composition of this portfolio by
  - increasing the weight on risky asset asset k by Δw_k,
  - and decreasing the weight of the riskless asset (by the same amount).
- **Question:** What effect does this have on the portfolio expected return and variance?
- The portfolio expected return is

$$E(R) = R_f + \sum_{n=1}^{N} w_n [E(R_n) - R_f].$$

Therefore the **change in expected return** is given by

$$\Delta E(R) = [E(R_k) - R_f] \Delta w_k.$$

Effects of Small Changes in Composition (cont'd)

- To determine the impact on variance, we compute (see next slide)

$$\frac{dV(R)}{dw_k} = 2Cov(R_k, R_P),$$

  where R_P denotes the return on the *initial* portfolio.

- Therefore, the **change in variance** is approximately given by

$$\Delta V(R) \approx 2Cov(R_k, R_P)\Delta w_k.$$

- **Remark:** The change in variance involves the covariance of asset k with the initial portfolio P, and not the variance of asset k.
  - When an asset is examined in isolation, the relevant measure of risk is the variance or standard deviation.
  - When an asset is examined as part of a portfolio, the relevant measure of risk is the **covariance** between the asset return and the return on the portfolio.

Optional Material: 'Marginal' Impact on Portfolio Variance

- The portfolio variance is

$$V(R) = \sum_{n=1}^{N} w_n^2 V(R_n) + \sum_{n=1}^{N} \sum_{m \neq n} w_n w_m Cov(R_n, R_m).$$

The terms that involve w_k are

$$w_k^2 V(R_k) + 2 \sum_{n \neq k} w_k w_n Cov(R_k, R_n).$$

- Therefore

$$\begin{aligned} \frac{dV(R)}{dw_k} &= 2w_k V(R_k) + 2 \sum_{n \neq k} w_n Cov(R_k, R_n) \\ &= 2w_k Cov(R_k, R_k) + 2 \sum_{n \neq k} w_n Cov(R_k, R_n) \\ &= 2Cov(R_k, \sum_{n=1}^{N} w_n R_n) = 2Cov(R_k, R_P). \end{aligned}$$

Can be positive or negative depending on the sign of $Cov(R_k, R_P)$.

8. Property of Portfolios on the CML

- Any portfolio P on the CML satisfies the following property:

$$\frac{E(R_k) - R_f}{Cov(R_k, R_P)} \text{ is equal across all assets } k = 1, \ldots, N.$$

- In words: the ratio of the change in expected return to the change in variance caused by a small change in portfolio weights is independent of the particular asset k towards which we tilt the portfolio.

- Logic of the proof (see next two slides)
  - Suppose, e.g., that the ratio is higher for asset n than for asset m.
  - Then it is possible to find a portfolio that beats portfolio P: by increasing the weight on n and decreasing the weight on m, we can reduce the variance, while keeping expected return constant.
  - Given that P is on the portfolio frontier, this is impossible. Therefore the ratio has to be equal across assets.

Optional Material: Proof of the Optimality Property

- We argue by contradiction. Suppose that

$$\frac{E(R_n) - R_f}{Cov(R_n, R_P)} > \frac{E(R_m) - R_f}{Cov(R_m, R_P)}.$$

- Consider an 'increase' in the weight of asset n, accompanied by an equal 'decrease' in the weight of the riskless asset, such that the expected return of portfolio increases by $dE > 0$.
- We need to 'increase' the weight of asset n by dw_n such that

$$[E(R_n) - R_f]dw_n = dE \quad \Rightarrow \quad dw_n = \frac{dE}{E(R_n) - R_f}.$$

- As a result the variance of the portfolio will 'increase' by

$$dV_n = 2Cov(R_n, R_P)dw_n = 2\frac{Cov(R_n, R_P)}{E(R_n) - R_f}dE.$$

Optional Material: Proof of the Optimality Property (cont'd)

- Consider also a 'decrease' in the weight of asset m, accompanied by an equal 'increase' in the weight of the riskless asset, such that the expected return of the portfolio decreases by dE.
- As a result, the variance of the portfolio will *'decrease'* by

$$dV_m = 2\frac{Cov(R_m, R_P)}{E(R_m) - R_f}dE.$$

- *Overall*, the expected return of the portfolio stays constant, while the variance changes by

$$dV_n - dV_m = 2\left(\frac{Cov(R_n, R_P)}{E(R_n) - R_f} - \frac{Cov(R_m, R_P)}{E(R_m) - R_f}\right)dE < 0.$$

- We have shown that we can form a portfolio that has the same expected return as P, with lower variance. This is a contradiction since P is a frontier portfolio.